Dirac-Kähler Theory and Massless Fields

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Three massless limits of the Dirac-Kähler theory are considered. It is shown that the Dirac-Kähler equation for massive particles can be represented as a result of the gauge-invariant mixture (topological interaction) of the above massless fields.

I. INTRODUCTION

In the 1980's, it was shown in [1] with the use of differential forms apparatus that the the Dirac-Kähler equation (DKE) is correct in describing Dirac particles (quarks) in the lattice formulation of QCD. After that the DKE has become to attract attention of many theorists. Note that the name "Dirac-Kähler equation" was introduced in [1] although the vector form of the DKE was discovered by Darwin [2]. His aim was to find an equation of motion of an electron that would be equivalent to the Dirac equation but without using spinors. Fundamental properties of the DKE were established by Kähler [3]. Later on, the DKE has been rediscovered in different mathematical formulations (see, e.g., ref. [4, pp.38,51] and references therein).

One should emphasize that up to now only massive DKE have been studied. The massless limit of the DKE has not been investigated in detail yet. The present paper is aimed at making up such a deficiency.

II. THE DIRAC-KÄHLER EQUATION

The DKE is equivalent to the following tensor system

$$\partial_\mu \psi_\mu + m\psi_0 = 0, \tag{1a}$$

$$\partial_\mu \tilde{\psi}_\mu + m\tilde{\psi}_0 = 0, \tag{1b}$$

$$\partial_\nu \psi_{[\mu\nu]} + \partial_\mu \psi_0 + m\psi_\mu = 0, \tag{1c}$$

$$\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\partial_\nu \psi_{[\alpha\beta]} + \partial_\mu \tilde{\psi}_0 + m\tilde{\psi}_\mu = 0, \tag{1d}$$

$$-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu + \varepsilon_{\mu\nu\alpha\beta}\partial_\alpha \tilde{\psi}_\beta + m\psi_{[\mu\nu]} = 0, \tag{1e}$$

where $\tilde{\psi}_\mu = \frac{1}{3!}\ \varepsilon_{\mu\nu\alpha\beta}\psi_{[\nu\,\alpha\beta]}$ is an axial vector, $\tilde{\psi}_0 = \frac{1}{4!}\ \varepsilon_{\mu\nu\alpha\beta}\psi_{[\mu\nu\alpha\beta]}$ is a pseudoscalar, and $\varepsilon_{\mu\nu\alpha\beta}$ is the Levi-Civita tensor ($\varepsilon_{1234} = -i$).

The set of equations (1) can be brought to the following form

$$(\Gamma_\mu \partial_\mu + m)\ \Psi = 0, \tag{2}$$

where Ψ is the 16-component wave function

$$\Psi \equiv \Psi_A : \psi_0, \tilde{\psi}_0, \psi_\mu, \tilde{\psi}_\mu, \psi_{[\mu\nu]}, \tag{3}$$

consisting of the Dirac-Kähler (DK) field components which form the full set of antisymmetric tensor fields in the space of dimension $d = 4$. The 16×16 matrices Γ_μ satisfy the anticommutation rules analogous to these for the Dirac matrices

$$\Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 2\ \delta_{\mu\nu}. \tag{4}$$

In respect to the theory of relativistic wave equations, the system of the DKE describes a particle with a single mass m, variable spin 0 or 1, and with double degeneration of states over an additional quantum number (internal parity). At the same time, the Lagrangian of the DK field is invariant under a transformation of the group of internal (dial) symmetry $SO\ (4,\ 2\ )$ [4, pp.28,35]. Group generators have the following form

$$\Gamma'_\mu\,, \quad \Gamma'_{[\mu}\,\Gamma'_{\nu]}\,, \quad \Gamma'_5\,\Gamma'_\mu\,, \quad \Gamma'_5\ . \tag{5}$$

Here Γ'_μ is second set of 16×16 matrices satisfying the Dirac matrix algebra and commuting with Γ_μ. The above properties of the symmetry are easily checked if one takes into account that in the so-called fermion basis (see, e.g., the book [4, p.72]) the matrices Γ_μ and Γ'_μ can be written as

$$\Gamma_\mu = I_4 \otimes \gamma_\mu \,, \qquad \Gamma'_\mu = \gamma_\mu \otimes I_4, \tag{6}$$

where γ_μ are the Dirac matrices, I_4 is the unit 4 by 4 matrix.

Internal ("dial") symmetry transformations relate with each other tensors of different ranks. Thereby, the theoretical group ground is established for association of a Dirac particle with the DK field. This particle ("geometric fermion") in addition to the spin $\frac{1}{2}$ has its inner degrees of freedom which are of space-time origin. For instance, if one turns on electromagnetic interaction in the standard way, i.e. by the replacement $\partial_\mu \to \partial_\mu - ieA_\mu$ (A_μ is an electromagnetic potential) then the DKE will have solutions equivalent to these for the Dirac equation because the matrices in both equations have the same algebraic properties.

Since we are going to study massless limits of the DKE it is convenient to transform the system (1) to another form replacing the common mass m by two new mass parameters, namely a parameter m_1 in (1a), (1b) and (1e), and a parameter m_2 in (1c) and (1d). One has the new system

$$\partial_\mu \psi_\mu + m_1 \psi_0 = 0, \tag{7a}$$

$$\partial_\mu \tilde{\psi}_\mu + m_1 \tilde{\psi}_0 = 0, \tag{7b}$$

$$\partial_\nu \psi_{[\mu\nu]} + \partial_\mu \psi_0 + m_2 \psi_\mu = 0, \tag{7c}$$

$$\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\partial_\nu \psi_{[\alpha\beta]} + \partial_\mu \tilde{\psi}_0 + m_2 \tilde{\psi}_\mu = 0, \tag{7d}$$

$$-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu + \varepsilon_{\mu\nu\alpha\beta}\partial_\alpha \tilde{\psi}_\beta + m_1 \psi_{[\mu\nu]} = 0. \tag{7e}$$

Matrix form of this system is as follows

$$(\Gamma_\mu \, \partial_\mu \; + \; m_1 \, P_1 \; + \; m_2 \, P_2) \; \Psi = 0, \tag{8}$$

where P_1 and P_2 are the projection operators with the properties

$$\begin{array}{l} P_1^2 = P_1, \quad P_2^2 = P_2, \quad P_1 + P_2 = 1, \quad P_1 P_2 = 0, \\ P_1 \Gamma_\mu + \Gamma_\mu P_1 = \Gamma_\mu, \quad P_2 \Gamma_\mu + \Gamma_\mu P_2 = \Gamma_\mu. \end{array} \tag{9}$$

A second order equation equivalent to the system (7) is

$$(\Box - m_1 m_2) \;\; \psi_A = 0. \tag{10}$$

This means that the system (7) at $m_1 \neq 0$ and $m_2 \neq 0$ describes a particle with a single mass $m = \sqrt{m_1 m_2}$, i.e. it does not differ from the usual DKE. But if any of the mass parameters (or both simultaneously) is equal to zero then the system (7) and the equation (8) correspond to the massless case. We proceed now to the study of this important case.

III. "ELECTROMAGNETIC" MASSLESS LIMIT

Let us suppose in (7) and (8) that

$$m_2 = 0. \tag{11}$$

If $m_1 \neq 0$, then without loosing of generality one can put $m_1 = 1$ so that the system (7) is now

$$\partial_\mu \psi_\mu + \psi_0 = 0, \tag{12a}$$

$$\partial_\mu \tilde{\psi}_\mu + \tilde{\psi}_0 = 0, \tag{12b}$$

$$\partial_\nu \psi_{[\mu\nu]} + \partial_\mu \psi_0 = 0, \tag{12c}$$

$$\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\partial_\nu \psi_{[\alpha\beta]} + \partial_\mu \tilde{\psi}_0 = 0, \tag{12d}$$

$$-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu + \varepsilon_{\mu\nu\alpha\beta}\partial_\alpha \tilde{\psi}_\beta + \psi_{[\mu\nu]} = 0, \tag{12e}$$

and correspondingly

$$(\Gamma_\mu \partial_\mu + P_1)\ \Psi = 0. \tag{13}$$

Note that the transition to the massless limit by means of introduction of projective operators into (2) is a typical procedure for fields with integer spins. For example, it is carried out when one wants to pass on to massless vector field (i.e. electromagnetic field) from the massive Duffin-Kemmer equation.

In order to establish the physical meaning of the system (12), we note, first of all, that the field functions Ψ_A obey the D'Alembert equation

$$\Box \Psi_A = 0, \tag{14}$$

i.e. this system describes a massless field. The vector $\psi_\mu(x)$ and the pseudovector $\tilde{\psi}_\mu(x)$ play role of potentials in (12), and the antisymmetric tensor $\psi_{[\mu\,\nu]}$ is an intensity tensor. Physical meaning of the scalar, $\psi_0(x)$, and pseudoscalar, $\tilde{\psi}_0(x)$, functions will be clarified later on.

The system (12) is invariant under gauge transformations of the potentials

$$\delta\psi_\mu\,(x) = \partial_\mu \lambda(x), \quad \delta\tilde{\psi}_\mu(x) = \partial_\mu \tilde{\lambda}(x), \tag{15}$$

where $\lambda(x)$ and $\tilde{\lambda}(x)$ satisfy the following conditions

$$\Box\ \lambda(x) = 0, \quad \Box\ \tilde{\lambda}(x) = 0. \tag{16}$$

To find independent physical states of the field system under consideration, we perform the Fourier transformation

$$\psi_\mu\ (x) \ = \ \int \psi_\mu\ (\underline{p})\ e^{i\,p\,x}\, d^3p + h.c., \tag{17a}$$

$$\tilde{\psi}_\mu\ (x) \ = \ \int \tilde{\psi}_\mu\ (\underline{p})\ e^{i\,p\,x}\, d^3p + h.c., \tag{17b}$$

$$\psi_0\ (x) \ = \ \int \psi_0\ (\underline{p})\ e^{i\,p\,x}\, d^3p + h.c., \tag{17c}$$

$$\tilde{\psi}_0\ (x) \ = \ \int \tilde{\psi}_0\ (\underline{p})\ e^{i\,p\,x}\, d^3p + h.c. \tag{17d}$$

Let us expand the amplitudes $\psi_\mu\ (\underline{p})$ and $\tilde{\psi}_\mu\ (\underline{p})$ over the complete basis $e^{(1)}_\mu$, $e^{(2)}_\mu$, p_μ, n_μ with properties [5]

$$\begin{aligned} e^{(i)}_\mu e^{(j)}_\mu = \delta_{i\,j},\ e^{(i)}_\mu p_\mu = 0,\ e^{(i)}_\mu n_\mu = 0, \\ p^2_\mu = 0,\ n^2_\mu = -1. \end{aligned} \tag{18}$$

Note that the basis (18) is not orthogonal because it contains an isotropic vector p_μ. The desired decompositions can be written as

$$\begin{aligned} \psi_\mu\ (\underline{p}) \ &= \ \sum_{i=1}^{2} a_i\, e^{(i)}_\mu \ + \ b\, p_\mu \ + \ c\, n_\mu\ , \\ \tilde{\psi}_\mu\ (\underline{p}) \ &= \ \sum_{i=1}^{2} \tilde{a}_i\, e^{(i)}_\mu \ + \ \tilde{b}\, p_\mu \ + \ \tilde{c}\, n_\mu. \end{aligned} \tag{19}$$

Now let us take into account that due to (16) the gauge functions $\lambda(x)$ and $\tilde{\lambda}(x)$ have the form analogous to (17c) and (17d)

$$\begin{aligned} \lambda\ (x) \ &= \ \int \lambda\ (\underline{p})\ e^{i\,p\,x}\, d^3p + h.c., \\ \tilde{\lambda}\ (x) \ &= \ \int \tilde{\lambda}\ (\underline{p})\ e^{i\,p\,x}\, d^3p + h.c, \end{aligned} \tag{20}$$

where $\lambda\,(\underline{p})$ and $\tilde{\lambda}\ (\underline{p})$ are arbitrary amplitudes. Inserting decompositions (17a)–(17d) in (15) we obtain gauge transformations for the amplitudes of the potentials

$$\begin{aligned} \delta\,\psi_\mu\ (\underline{p}) = i\lambda\ (\underline{p})\ p_\mu, \\ \delta\,\tilde{\psi}_\mu\ (\underline{p}) = i\,\tilde{\lambda}\ (\underline{p})\ p_\mu, \end{aligned} \tag{21}$$

which mean that the amplitudes $\psi_\mu(\underline{p})$ and $\tilde{\psi}_\mu\ (\underline{p})$ are defined up to unessential terms $i\,\lambda\ (\underline{p})\ p_\mu$ and $i\,\tilde{\lambda}\ (\underline{p})\ p_\mu$, respectively. Role of such terms in the decompositions (19) play $b\,p_\mu$ and $\tilde{b}\,p_\mu$. Omitting them we obtain for the amplitudes $\psi_\mu(\underline{p})$ and $\tilde{\psi}_\mu\ (\underline{p})$

$$
\begin{aligned}
\psi_\mu\ (\underline{p}) &= \sum_{i=1}^{2} a_i\, e_\mu^{(i)}\ +\ c\, n_\mu\ , \\
\tilde{\psi}_\mu\ (\underline{p}) &= \sum_{i=1}^{2} \tilde{a}_i\, e_\mu^{(i)}\ +\ \tilde{c}\, n_\mu .
\end{aligned}
\tag{22}
$$

Note that longitudinal oscillations (degrees of freedom) are absent in (22).

Scalar degrees of freedom are eliminated at the second quantization procedure when the equations (12a) and (12b) for quantized field are formulated in the form of conditions imposed on wave functions Ψ_{phys} in the state space

$$
\begin{aligned}
&\left(\partial_\mu \hat{\psi}_\mu(x) + \hat{\psi}_0(x)\right)_+ \Psi_{phys} = 0, \\
&\left(\partial_\mu \hat{\tilde{\psi}}_\mu(x) + \hat{\tilde{\psi}}_0(x)\right)_+ \Psi_{phys} = 0,
\end{aligned}
\tag{23}
$$

where the index " + " means that the corresponding operator contains the positive-frequency part only. Keeping in mind the relations (17), (18) and (22) we obtain from (23)

$$
\begin{aligned}
&\left(\int \omega \left(\hat{d} - \hat{c}\right) e^{ipx} d^3p\right)\ \Psi_{phys} = 0, \\
&\left(\int \omega \left(\hat{\tilde{d}} - \hat{\tilde{c}}\right) e^{ipx} d^3p\right)\ \Psi_{phys} = 0,
\end{aligned}
\tag{24}
$$

where

$$
d = \frac{\psi_0\ (p)}{\omega}\ , \qquad \tilde{d} = \frac{\tilde{\psi}_0\ (p)}{\omega}
\tag{25}
$$

play a role of the amplitudes of the scalar fields ψ_0 and $\tilde{\psi}_0$. It is follows from (24) that for all p, the function Ψ_{phys} has to satisfy the following conditions

$$
\left(\hat{d} - \hat{c}\right)\ \Psi_{phys} = 0,\ \left(\hat{\tilde{d}} - \hat{\tilde{c}}\right)\ \Psi_{phys} = 0,
\tag{26}
$$

A standard procedure used to eliminate longitudinal and scalar oscillations at quantization of electromagnetic field [6,pp.56,68] leads to relations [7]

$$
\begin{aligned}
&\left(\Psi_{phys}, \left(\hat{d}^+ \hat{d} + \hat{c}^+ \hat{c}\right) \Psi_{phys}\right) = 0, \\
&\left(\Psi_{phys}, \left(\hat{\tilde{d}}^+\ \hat{\tilde{d}} + \hat{\tilde{c}}^+\ \hat{\tilde{c}}\right) \Psi_{phys}\right) = 0.
\end{aligned}
\tag{27}
$$

Due to (27) the mean values disappear in the part of the energy operator that contains scalar oscillations of both types. Therefore, the system (12) describes a massless vector field with double degeneration of states. A special case of such a field is the usual electromagnetic field. As is follows from the previous analysis, there are no physical states corresponding to the scalar and pseudoscalar functions $\psi_0(x)$ and $\tilde{\psi}_0(x)$. These functions serve as gauge fields ("ghosts").

IV. NOTOPH (KALB-RAMOND FIELD)

Let us consider the following case of the system (7):

$$
m_1 = 0, \quad m_2 = 1.
\tag{28}
$$

In this case one has for (7)

$$\partial_\mu \psi_\mu = 0, \tag{29a}$$

$$\partial_\mu \tilde{\psi}_\mu = 0, \tag{29b}$$

$$\partial_\nu \psi_{[\mu\nu]} + \partial_\mu \psi_0 + \psi_\mu = 0, \tag{29c}$$

$$\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\partial_\nu \psi_{[\alpha\beta]} + \partial_\mu \tilde{\psi}_0 + \tilde{\psi}_\mu = 0, \tag{29d}$$

$$-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu + \varepsilon_{\mu\nu\alpha\beta}\partial_\alpha \tilde{\psi}_\beta = 0. \tag{29e}$$

Here ψ_0, $\tilde{\psi}_0$ and $\psi_{[\mu\nu]}$ serve as potentials and the vectors ψ_μ and $\tilde{\psi}_\mu$ serve as intensities. The equations (29c) and (29d) are definitions of intensities via potentials and the equations (29a), (29b), and (29e) are equations of motion.

The matrix form of the system (29) is

$$(\Gamma_\mu \partial_\mu \; + \; P_2)\; \Psi = 0. \tag{30}$$

Using either tensor or matrix formulation of the field system under consideration, one can easily show that all components of the wave function obey the D'Alembert equation (14), i.e. again we deal with the massless field.

Further discussion becomes to be more convenient if one introduces an auxiliary tensor $\tilde{\psi}_{[\mu\nu]}$

$$\tilde{\psi}_{[\mu\nu]} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\,\psi_{[\alpha\beta]}\;. \tag{31}$$

One has for (29d)

$$\partial_\nu \tilde{\psi}_{[\mu\nu]} \; + \; \partial_\mu \tilde{\psi}_0 \; + \; \tilde{\psi}_\mu = 0. \tag{32}$$

This equation will be used below together with (29d).

In order to pass on to the momentum representation, we write down the scalar potentials in form of (17c), (17d) and tensor potentials in the following form

$$\begin{aligned} \psi_{[\mu\nu]}\,(x) &= \int \psi_{[\mu\nu]}\,(\underline{p})\; e^{ipx}\, d^3 p + h.c., \\ \tilde{\psi}_{[\mu\nu]}\,(x) &= \int \tilde{\psi}_{[\mu\nu]}\,(\underline{p})\; e^{ipx}\, d^3 p + h.c. \end{aligned} \tag{33}$$

We expand now the amplitudes $\psi_{[\mu\nu]}\,(\underline{p})$ and $\tilde{\psi}_{[\mu\nu]}\,(\underline{p})$ over the complete basis (18)

$$\begin{aligned} \psi_{[\mu\nu]}\,(\underline{p}) \; &= \; f\left(e^{(1)}_\mu e^{(2)}_\nu - e^{(1)}_\nu e^{(2)}_\mu\right) + \\ &+ \sum_{i=1}^{2} g_i \left(e^{(i)}_\mu p_\nu \; - \; e^{(i)}_\nu p_\mu\right) + \\ &+ \sum_{i=1}^{2} h_i \left(e^{(i)}_\mu n_\nu \; - \; e^{(i)}_\nu n_\mu\right) + \\ &+ e\,(p_\mu n_\nu \; - \; p_\nu n_\mu)\,, \end{aligned} \tag{34a}$$

$$\begin{aligned} \tilde{\psi}_{[\mu\nu]}\,(\underline{p}) \; &= \; \tilde{f}\left(e^{(1)}_\mu e^{(2)}_\nu \; - \; e^{(1)}_\nu e^{(2)}_\mu\right) + \\ &+ \sum_{i=1}^{2} \tilde{g}_i \left(e^{(i)}_\mu p_\nu \; - \; e^{(i)}_\nu p_\mu\right) + \\ &+ \sum_{i=1}^{2} \tilde{h}_i \left(e^{(i)}_\mu n_\nu - e^{(i)}_\nu n_\mu\right) + \\ &+ \tilde{e}\,(p_\mu n_\nu - p_\nu n_\mu)\,. \end{aligned} \tag{34b}$$

Further we take into account that the system (29) is invariant at the gauge transformations

$$\begin{aligned} \delta\,\psi_{[\mu\nu]}\,(x) \; &= \; \partial_\mu \lambda_\nu\,(x) \; - \; \partial_\nu \lambda_\mu\,(x) + \\ &+ \varepsilon_{\mu\nu\alpha\beta}\,\partial_\alpha \tilde{\lambda}_\beta\,(x)\,, \end{aligned} \tag{35}$$

where the gauge finctions $\lambda_\mu(x)$ and $\tilde{\lambda}_\mu(x)$ satisfy the conditions

$$\Box\lambda_\mu - \partial_\mu\partial_\nu\lambda_\nu = 0, \quad \Box\tilde{\lambda}_\mu - \partial_\mu\partial_\nu\tilde{\lambda}_\nu = 0. \tag{36}$$

Keeping in mind the symmetry between the tensors $\psi_{[\mu\nu]}$ and $\tilde{\psi}_{[\mu\nu]}$ one can replace (35) by

$$\begin{aligned} \delta\,\psi_{[\mu\nu]}(x) &= \partial_\mu\,\lambda_\nu(x) - \partial_\nu\,\lambda_\mu(x)\,, \\ \delta\,\tilde{\psi}_{[\mu\nu]}(x) &= \partial_\mu\,\tilde{\lambda}_\nu(x) - \partial_\nu\,\tilde{\lambda}_\mu(x)\,, \end{aligned} \tag{37}$$

where $\lambda_\mu(x)$ and $\tilde{\lambda}_\mu(x)$ still satisfy (36). As in the case of the equation (14), the solutions of (36) are superpositions of the plane waves

$$\lambda_\mu(x) = \int \lambda_\mu(\underline{p})\, e^{ipx}\, d^3p + h.c., \tag{38}$$

$$\tilde{\lambda}_\mu(x) = \int \tilde{\lambda}_\mu(\underline{p})\, e^{ipx}\, d^3p + h.c., \tag{39}$$

with the only difference is that now the amplitudes $\lambda_\mu(\underline{p})$ and $\tilde{\lambda}_\mu(\underline{p})$ being expanded over the basis (18) have the structure

$$\begin{aligned} \lambda_\mu(\underline{p}) &= \sum_{i=1}^{2} \alpha_i\, e_\mu^{(i)} + \beta\, p_\mu\,, \\ \tilde{\lambda}_\mu(\underline{p}) &= \sum_{i=1}^{2} \tilde{\alpha}_i\, e_\mu^{(i)} + \tilde{\beta}\, p_\mu, \end{aligned} \tag{40}$$

which does not contain terms with n_μ (due to terms $\partial_\mu\,\partial_\nu\,\lambda_\nu(x)$ and $\partial_\mu\,\partial_\nu\,\tilde{\lambda}_\nu(x)$ in (36)). Inserting (33) and (38)–(40) in (37) we obtain the following form of the gauge transformations for the potentials $\psi_{[\mu\nu]}$ and $\tilde{\psi}_{[\mu\nu]}$

$$\begin{aligned} \delta\,\psi_{[\mu\nu]}(\underline{p}) &= i\sum_{i=1}^{2} \alpha_i \left(e_\mu^{(i)}\, p_\nu - e_\nu^{(i)}\, p_\mu \right), \\ \delta\,\tilde{\psi}_{[\mu\nu]}(\underline{p}) &= i\sum_{i=1}^{2} \tilde{\alpha}_i \left(e_\mu^{(i)}\, p_\nu - e_\nu^{(i)}\, p_\mu \right). \end{aligned} \tag{41}$$

The expressions (41) show that terms containing g_i and $\tilde{g}_i$ in (34a) and (34b) are inessential. Therefore, one can eliminate them by an appropriate choice of the parameters α_i and $\tilde{\alpha}_i$ ($\alpha_i = -i\,g_i$ and $\tilde{\alpha}_i = -i\,\tilde{g}_i$), i.e. if one puts

$$\psi_{[23]} = \psi_{[31]} = \tilde{\psi}_{[23]} = \tilde{\psi}_{[31]} = 0. \tag{42}$$

Using the definition (31) for the tensor $\tilde{\psi}_{[\mu\nu]}$ and a relation following from this definition

$$\psi_{[\alpha\beta]} = -\frac{1}{2}\,\varepsilon_{\alpha\beta\mu\nu}\,\tilde{\psi}_{[\mu\nu]}, \tag{43}$$

we obtain from (42)

$$\psi_{[14]} = \psi_{[24]} = \tilde{\psi}_{[14]} = \tilde{\psi}_{[24]} = 0. \tag{44}$$

As a result, the decompositions (34a) and (34b) take the form

$$\begin{aligned} \psi_{[\mu\nu]}(\underline{p}) &= f\left(e_\mu^{(1)} e_\nu^{(2)} - e_\nu^{(1)} e_\mu^{(2)} \right) + e\,(p_\mu n_\nu - p_\nu n_\mu)\,, \\ \tilde{\psi}_{[\mu\nu]}(\underline{p}) &= \tilde{f}\left(e_\mu^{(1)} e_\nu^{(2)} - e_\nu^{(1)} e_\mu^{(2)} \right) + \tilde{e}\,(p_\mu n_\nu - p_\nu n_\mu)\,. \end{aligned} \tag{45}$$

The expressions (45) show that the tensor-potential $\psi_{[\mu\nu]}$ contains only two independent components either corresponding to the state of the massless spin-1 field with the longitudinal polarization. In the literature, such a field is known as "notoph" [5] or "Kalb-Ramond field" [8]. Because the system (29) contains also the potentials $\psi_0(x)$ and $\tilde{\psi}_0(x)$, we conclude that this system (or matrix equation (30) that is equivalent to it) describes the Kalb-Ramond field and the massless scalar field with a doubled set of states degenerated over an additional quantum number.

The massless field systems (12) and (29) like the DKE for a massive particle have an internal symmetry. Making the use of the matrix form of these systems (13) and (30) and explicit form of the matrices Γ_μ, Γ'_μ, P_1, and P_2 one can show that the system symmetry narrows up to the group $SO(3,1)$ of which generators are determined by the matrices $\Gamma'_{[i}\Gamma'_{k]}$ and $\Gamma'_5\Gamma'_k$.

V. MASSLESS "FERMION" LIMIT

The DKE is an equation describing a free massive Dirac particle with mass m. Therefore, a natural way to pass on to the massless limit in (7) is to put there $m_1 = m_2 = 0$. We will denote such a transition as "fermion" limit. As applied to the system of the tensor equations (7), this transition leads the systems

$$\partial_\nu \psi_{[\mu\nu]} + \partial_\mu \psi_0 = 0, \tag{46a}$$

$$\frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} \partial_\nu \psi_{[\alpha\beta]} + \partial_\mu \tilde{\psi}_0 = 0, \tag{46b}$$

$$\partial_\mu \psi_\mu = 0, \tag{46c}$$

$$\partial_\mu \tilde{\psi}_\mu = 0, \tag{46d}$$

$$-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu + \varepsilon_{\mu\nu\alpha\beta} \partial_\alpha \tilde{\psi}_\beta = 0. \tag{46e}$$

This system disintegrates over the Lorenz group into the two subsystems (46a),(46b) and (46c)–(46e).

It is evident that each of these subsystems can be written down in a matrix form with field functions $U(x)$ and $V(x)$ composed from ψ_0, $\tilde{\psi}_0$, $\psi_{[\mu\nu]}$, and ψ_μ, $\tilde{\psi}_\mu$, respectively. In addition, if relatively to the Lorenz group, the function $U(x)$ is transformed as a direct sum T of presentations for a bivector, a scalar, and a pseudoscalar then an equation for $U(x)$ is transformed as a direct sum R of presentations for a vector and a pseudovector. For the function $V(x)$ and an equation for it the presentations T and R are interchanged. This means that for each of the subsystems (46a),(46b) and (46c)–(46e) taken separately, a massless analog is absent (a field function for a massive equation and the equation itself are transformed over the same presentation of the Lorenz group). Therefore, only bilinear forms like $\bar{U}(x)\, \beta_\mu\, \partial_\mu V(x)$ and $\bar{V}(x)\, \beta_\mu\, \partial_\mu U(x)$ are Lorenz-invariant. Here β_μ are 8 by 8 matrices of equations for $U(x)$ and $V(x)$. Thus, although the subsystems (46a),(46b) and (46c)–(46e) are independent algebraically the Lagrangian formulation for them is impossible. The requirements of the Lagrangian formulation of a theory and existence of its massive analog lead to the need for combined consideration of the subsystems (46a),(46b) and (46c)–(46e). This circumstance is of crucial importance because the energy-momentum density turns out to be zero for such a massless system.

Indeed, the Lagrangian for the system (46) can be written as

$$\begin{aligned} L \; = \; & - \psi_\mu \partial_\mu \psi_0 - \frac{1}{2} \psi_{[\mu\nu]} (\partial_\mu \psi_\nu - \partial_\nu \psi_\mu) + \\ & + \tilde{\psi}_\mu \partial_\mu \tilde{\psi}_0 + \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} \psi_{[\mu\nu]} \partial_\alpha \tilde{\psi}_\beta. \end{aligned} \tag{47}$$

Inserting (47) in the expression for the energy-momentum

$$T_{\mu\nu} = \frac{\partial L}{\partial \left(\frac{\partial \Psi_A}{\partial x_\mu} \right)} \frac{\partial \Psi_A}{\partial x_\nu} - \delta_{\mu\nu} L, \tag{48}$$

using further the obtained formula and equations (46) and keeping in mind that terms like full divergency can be omitted, one finally has

$$T_{\mu\nu} = 0. \tag{49}$$

Hence, the fermion massless limit of the DKE treated as equations describing a classical boson field (a particle with variable spin 0,1) leads to zero energy-momentum density.

VI. GAUGE-INVARIANT MIXING OF MASSLESS FIELDS

In papers [8, 9] a non-Higgs mechanism to generate masses was proposed. The method relies on gauge-invariant mixing (the topological interaction) an electromagnetic field and a massless vector field with the zero helicity (the so-called $\hat{B} \wedge \hat{F}$-theory). The final result of that theory is the Duffin-Kemmer equation for a massive spin-1 particle.

Such an approach is very important for the string theory (see [10–13] and references there in). It is, therefore, important to generalize this approach for the case of massless systems of the DK type involving the complete set of antisymmetric tensor fields in the space with dimension $d = 4$.

At first, we will proceed from the matrix formulation (13) and (30) of the systems (12) and (29). We replace the notation Ψ in (13) by Φ, φ_0, $\tilde{\varphi}_0$, φ_μ, $\tilde{\varphi}_\mu$, $\varphi_{[\mu\,\nu]}$. Now the Lagrangian of the matrix equations (13) and (30) takes the form

$$L_0 = -\overline{\Phi}\,(\Gamma_\mu\,\partial_\mu \ + \ P_1)\,\Phi \ - \ \overline{\Psi}\,(\Gamma_\mu\,\partial_\mu \ + \ P_2)\,\Psi \ , \tag{50}$$

where $\overline{\Phi} = \Phi^+\,\Gamma_4\,\Gamma'_4\,,\ \ \overline{\Psi} \ = \ \Psi^+\,\Gamma_4\,\Gamma'_4$. We add now to L_0 a term

$$L_{int} = -\,m\,\overline{\Phi}\,P_2\,\Psi - m\,\overline{\Psi}\,P_1\,\Phi, \tag{51}$$

which does not destroy the gauge invariance of the Lagrangian (50) relatively to transformations (15), (16), (35), and (36). One can obtain then the following matrix equations from the total Lagrangian $L = L_0 + L_{int}$

$$\Gamma_\mu\,\partial_\mu\,\Phi + P_1\,\Phi + m\,P_2\,\Psi \ = \ 0, \tag{52}$$
$$\Gamma_\mu\,\partial_\mu\,\Psi + P_2\,\Psi + m\,P_1\Phi \ = \ 0. \tag{53}$$

Multiplying (52) from the left by the matrix P_1 and using relations (9), we obtain an equation

$$\Gamma_\mu\,\partial_\mu\,P_1\,\Phi + m\,P_2\,\Psi = 0. \tag{54}$$

Analogously, multiplying (53) from the left by the matrix P_1 one has

$$\Gamma_\mu\,\partial_\mu\,P_2\,\Psi + m\,P_1\,\Phi = 0. \tag{55}$$

If one puts together (54) and (55) and introduces a notation

$$\Psi' = P_1\Phi + P_2\,\Psi = \Big(\varphi_0\,,\ \ \tilde{\varphi}_0\,,\ \ \psi_\mu\,,\ \ \tilde{\psi}_\mu\,,\ \ \varphi_{[\mu\,\nu]}\Big)\,, \tag{56}$$

one arrives at the following equation

$$(\Gamma_\mu\partial_\mu + m)\ \Psi' = 0, \tag{57}$$

that ut to the notation of the components of the wave function coincides with the DKE (2) and (3).

Therefore, the DKE for massive particles can be represented as a result of a gauge-invariant mixture of two massless systems, namely dial symmetric generalizations of electromagnetic field and the Kalb-Ramond field (notoph). All known in the literature constructions of the analogous mechanism for tensor fields (an abelian case) of different ranks in the $d = 4$ space are particular cases of the considered approach.

Now let us consider the tensor field system (46) together with another system of the same kind

$$\begin{aligned}
\partial_\nu\varphi_{[\mu\,\nu]} + \partial_\mu\varphi_0 = 0,\\
\frac{1}{2}\varepsilon_{\mu\nu\,\alpha\,\beta}\partial_\nu\varphi_{[\alpha\,\beta]} + \partial_\mu\tilde{\varphi}_0 = 0,\\
\partial_\mu\varphi_\mu = 0,\\
\partial_\mu\tilde{\varphi}_\mu = 0,\\
-\partial_\mu\varphi_\nu + \partial_\nu\varphi_\mu + \varepsilon_{\mu\,\nu\,\alpha\,\beta}\partial_\alpha\tilde{\varphi}_\beta = 0.
\end{aligned} \tag{58}$$

The Lagrangian of these systems takes the structure

$$L_0 = L_{01} + L_{02}, \tag{59}$$

where Lagrangians L_{01} and L_{02} have the form (47). The topological interaction of both systems can be described by the Lagrangian

$$\begin{aligned}
L_{int} \ = \ m\ \Big(a\varphi_0\psi_0 + b\tilde{\varphi}_0\tilde{\psi}_0 + c\varphi_\mu\psi_\mu +\\
+\ d\tilde{\varphi}_\mu\tilde{\psi}_\mu + e\varphi_{[\mu\,\nu]}\psi_{[\mu\,\nu]}\Big)\,.
\end{aligned} \tag{60}$$

From the total Lagrangian $L = L_0 + L_{int}$ one can obtain the following equations

$$\begin{aligned}
\partial_\mu \varphi_\mu + am\psi_0 &= 0,\\
\partial_\mu \tilde{\varphi}_\mu - bm\tilde{\psi}_0 &= 0,\\
\partial_\nu \varphi_{[\mu\,\nu]} + \partial_\mu \varphi_0 - cm\psi_\mu &= 0,\\
\frac{1}{2}\varepsilon_{\mu\,\nu\,\alpha\,\beta}\partial_\nu \varphi_{[\alpha\,\beta]} + \partial_\mu \tilde{\varphi}_0 + dm\tilde{\psi}_\mu &= 0,\\
-\partial_\mu \varphi_\nu + \partial_\nu \varphi_\mu + \varepsilon_{\mu\,\nu\,\alpha\,\beta}\partial_\alpha \tilde{\varphi}_\beta + 2em\psi_{[\mu\,\nu]} &= 0,\\
\partial_\nu \psi_\mu + am\varphi_0 &= 0,\\
\partial_\mu \tilde{\psi}_\mu - bm\tilde{\varphi}_0 &= 0,\\
\partial_\nu \psi_{[\mu\,\nu]} + \partial_\mu \psi_0 - cm\varphi_\mu &= 0,\\
\frac{1}{2}\varepsilon_{\mu\,\nu\,\alpha\,\beta}\partial_\nu \psi_{[\alpha\,\beta]} + \partial_\mu \tilde{\psi}_0 + dm\tilde{\varphi}_\mu &= 0,\\
-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu + \varepsilon_{\mu\,\nu\,\alpha\,\beta}\partial_\alpha \tilde{\psi}_\beta + 2em\varphi_{[\mu\,\nu]} &= 0.
\end{aligned} \tag{61}$$

Then we can make transition from the equations to the two systems

$$\begin{aligned}
\partial_\mu \Lambda_\mu + am\Lambda_0 &= 0,\\
\partial_\mu \tilde{\Lambda}_\mu - bm\tilde{\Lambda}_0 &= 0,\\
\partial_\nu \Lambda_{[\mu\,\nu]} + \partial_\mu \Lambda_0 - cm\Lambda_\mu &= 0,\\
\frac{1}{2}\varepsilon_{\mu\,\nu\,\alpha\,\beta}\partial_\nu \Lambda_{[\alpha\,\beta]} + \partial_\mu \tilde{\Lambda}_0 + dm\tilde{\Lambda}_\mu &= 0,\\
-\partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu + \varepsilon_{\mu\,\nu\,\alpha\,\beta}\partial_\alpha \tilde{\Lambda}_\beta + 2em\Lambda_{[\mu\,\nu]} &= 0
\end{aligned} \tag{62}$$

and

$$\begin{aligned}
\partial_\mu \Omega_\mu - am\Omega_0 &= 0,\\
\partial_\mu \tilde{\Omega}_\mu + bm\tilde{\Omega}_0 &= 0,\\
\partial_\nu \Omega_{[\mu\,\nu]} + \partial_\mu \Omega_0 + cm\Omega_\mu &= 0,\\
\frac{1}{2}\varepsilon_{\mu\,\nu\,\alpha\,\beta}\partial_\nu \Omega_{[\alpha\,\beta]} + \partial_\mu \tilde{\Omega}_0 - dm\tilde{\Omega}_\mu &= 0,\\
-\partial_\mu \Omega_\nu + \partial_\nu \Omega_\mu + \varepsilon_{\mu\,\nu\,\alpha\,\beta}\partial_\alpha \tilde{\Omega}_\beta - 2em\Omega_{[\mu\,\nu]} &= 0,
\end{aligned} \tag{63}$$

where

$$\begin{array}{l}
\Lambda_0 = \varphi_0 + \psi_0, \quad \tilde{\Lambda}_0 = \tilde{\varphi}_0 + \tilde{\psi}_0, \quad \Lambda_\mu = \varphi_\mu + \psi_\mu,\\
\tilde{\Lambda}_\mu = \tilde{\varphi}_\mu + \tilde{\psi}_\mu, \quad \Lambda_{[\mu\,\nu]} = \varphi_{[\mu\,\nu]} + \psi_{[\mu\,\nu]},\\
\Omega_0 = \varphi_0 - \psi_0, \quad \tilde{\Omega}_0 = \tilde{\varphi}_0 - \tilde{\psi}_0, \quad \Omega_\mu = \varphi_\mu - \psi_\mu,\\
\tilde{\Omega}_\mu = \tilde{\varphi}_\mu - \tilde{\psi}_\mu, \quad \Omega_{[\mu\,\nu]} = \varphi_{[\mu\,\nu]} - \psi_{[\mu\,\nu]}.
\end{array} \tag{64}$$

In the case of choosing

$$a = d = 1, \quad b = c = -1, \quad e = \frac{1}{2} \tag{65}$$

we obtain two types of the DKE.

The matrix form of the Lagrangian of the equations (46) and (58) is

$$L_0 = -\bar{\Phi}\Gamma_\mu \partial_\mu \Phi - \bar{\Psi}\Gamma_\mu \partial_\mu \Psi. \tag{66}$$

We add to (66) the term

$$L_{int} = -m\bar{\Phi}\Psi - m\bar{\Psi}\Phi. \tag{67}$$

As a result, we obtain the matrix equations

$$\Gamma_\mu \partial_\mu \Phi + m\Psi = 0, \tag{68}$$
$$\Gamma_\mu \partial_\mu \Psi + m\Phi = 0. \tag{69}$$

Putting together (68) and (69) and introducing the notation

$$\Lambda = \Phi + \Psi, \quad \Omega = \Phi - \Psi, \tag{70}$$

one arrives at the equations

$$\left(\Gamma_\mu \partial_\mu + m\right) \Lambda = 0, \tag{71}$$
$$\left(\Gamma_\mu \partial_\mu - m\right) \Omega = 0, \tag{72}$$

which are the matrix analogs of the tensor systems (62)–(65).

VII. CONCLUSION

We have investigated there massless limits of the DKE. It is shown that first of the limits leads to a two-potential formulation of the electrodynamics in the Feynman gauge. Second one gives a generalized description of the Ogievetsky-Polubarinov notoph. Third limit corresponds to the massless field with zero energy density. The method to generate masses through the gauge-invariant mixing (the topological interaction) of massless fields ($\hat{B} \wedge \hat{F}$-theory) is generalized for the case of above massless systems of the DK type. As a result, the DKE of two types for particles with masses are found. One of these equations after quantization can obey Bose-Einstein statistics and second one can obey Dirac-Fermi statistics [14, 15]. This means that from the topological interaction of the massless DK fields we can proceed to the tensor and Dirac fields with masses.

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